Homework 1
Algebra

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Lemma 0.1 (for Exercise 1a)**.** *Let R be a commutative ring with an ideal I and let M be an R-module. Then*

$$M \otimes_R R/I \cong M/IM$$

Proof. Define $\phi : M \times R/I \to M/IM$ by $(m, \overline{x}) \mapsto \overline{xm}$ where $m \in M, x \in R$, and $\overline{x} = x + I \in R/I$. We claim that this is well-defined. If $(m', \overline{x'}) = (m, \overline{x})$, then $m = m'$ and

$$\overline{x} = \overline{x'} \implies x - x' \in I \implies (x - x')m \in IM \implies \overline{xm} = \overline{x'm}$$

Thus ϕ is well-defined. Now we show that ϕ is R-bilinear. Let $x, x', y \in R$ and $m, m' \in M$. Then

$$\begin{aligned} \phi(m + m', \overline{x}) &= \overline{x(m + m')} = \overline{xm} + \overline{xm'} = \phi(m, \overline{x}) + \phi(m, \overline{x'}) \\ \phi(m, \overline{x} + \overline{x'} &= \overline{(x + x')m} = \overline{xm} + \overline{x'm} = \phi(m, \overline{x}) + \phi(m, \overline{x'}) \\ \phi(ym, \overline{x}) &= \overline{yxm} = y\overline{xm} = y\phi(m, \overline{x}) \\ \phi(m, y\overline{x}) &= \phi(m, \overline{yx}) = \overline{yxm} = y\overline{xm} = y\phi(m, \overline{x}) \end{aligned}$$

Then by the universal property of the tensor product, there exists an R-module homomorphism $\widetilde{\phi} : M \otimes_R R/I \to M/IM$ so that the following diagram commutes.

$$\begin{array}{ccc} M \times R/I & \xrightarrow{\otimes} & M \otimes_R R/I \\ & \overset{\phi}{\searrow} & \downarrow \widetilde{\phi} \\ & & M/IM \end{array}$$

that is, $\widetilde{\phi}(m \otimes \overline{x}) = \phi(m, \overline{x}) = \overline{xm}$. Define $\psi : M \to M \otimes_R R/I$ by $m \mapsto m \otimes \overline{1}$. Then ψ is an R-module homomorphism because

$$\begin{aligned} \psi(m + m') &= (m + m') \otimes \overline{1} = m \otimes \overline{1} + m' \otimes \overline{1} = \psi(m) + \psi(m') \\ \psi(xm) &= (xm) \otimes \overline{1} = x(m \otimes \overline{1}) = x\psi(m) \end{aligned}$$

We claim that $IM \subset \ker \psi$. Every element of IM is of the form am where $a \in I$ and $m \in M$. Then $\overline{a} = I = 0$ in R/I, so

$$\psi(am) = (am) \otimes \overline{1} = m \otimes \overline{a} = m \otimes 0 = 0$$

Thus the map $\widetilde{\psi} : M/IM \to M \otimes_R R/I$ given by $\overline{m} \mapsto m \otimes \overline{1}$ is a well-defined R-module homomorphism. Finally, we claim that $\widetilde{\psi}$ is an inverse to $\widetilde{\phi}$.

$$\widetilde{\phi}\widetilde{\psi}(\overline{m}) = \widetilde{\phi}(m \otimes \overline{1}) = \overline{1m} = \overline{m}$$
$$\widetilde{\psi}\widetilde{\phi}(m \otimes \overline{x}) = \widetilde{\psi}(\overline{xm}) = (xm) \otimes \overline{1} = m \otimes \overline{x}$$

Note that it is enough to check that $\widetilde{\psi}\widetilde{\phi} = \mathrm{Id}$ on simple tensors since the simple tensors generate $M \otimes_R R/I$. Thus $\widetilde{\psi}$ and $\widetilde{\phi}$ are inverses, so $\widetilde{\phi}$ is an isomorphism of R-modules. □

Proposition 0.2 (Exercise 1a)**.** *Let A be an abelian group and $n > 0$ an integer. Then*

$$A \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong A/nA$$

Proof. Take $R = \mathbb{Z}$, $M = A$, and $I = n\mathbb{Z}$ and apply the previous lemma. Note that $nA = (n\mathbb{Z})A$.

□

Lemma 0.3 (for Exericse 1b)**.** *Let $m, n \in \mathbb{Z}$. The order of $\overline{n}$ in $\mathbb{Z}/m\mathbb{Z}$ is $\frac{m}{\gcd(n,m)}$. Consequently, $n(\mathbb{Z}/m\mathbb{Z})$ is cyclic of order $\frac{m}{\gcd(n,m)}$.*

Proof. The order of $\overline{n}$ is the smallest positive integer multiple of n that divides m which is $\frac{m}{\gcd(n,m)}$. Then note that $n + m\mathbb{Z}$ generates $n(\mathbb{Z}/m\mathbb{Z})$, so the orders match. □

Lemma 0.4 (for Exercise 1b)**.** *Let $m, n \in \mathbb{Z}$ and $d = \gcd(m, n)$. Then*

$$(\mathbb{Z}/m\mathbb{Z})/n(\mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$$

Proof. The quotient of a cyclic group is cyclic. The previous lemma gives the order of $n/(\mathbb{Z}/m\mathbb{Z})$ as $\frac{m}{\gcd(n,m)}$. Hence $(\mathbb{Z}/m\mathbb{Z})/n(\mathbb{Z}/m\mathbb{Z})$ is cyclic of order

$$\frac{m}{\left(\frac{m}{\gcd(n,m)}\right)} = \gcd(n, m)$$

□

Proposition 0.5 (Exercise 1b)**.** *Let $m, n \in \mathbb{Z}$ and $d = \gcd(m, n)$. Then*

$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$$

Proof. Using Exercise 1(a) with $A = \mathbb{Z}/m\mathbb{Z}$, and then applying the above lemma,

$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong (\mathbb{Z}/m\mathbb{Z})/n(\mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$$

□

Lemma 0.6 (for Exercise 2a)**.** *Let A be a nonzero finitely generated torsion abelian group. Then $A \otimes_{\mathbb{Z}} A \neq 0$.*

Proof. By the classification of finitely generated abelian groups,

$$A \cong \bigoplus_{i=1}^{N} \mathbb{Z}/k_i\mathbb{Z}$$

where $k_i \in \{2, 3, \dots\}$ so that $k_1|k_2|k_2|\dots|k_N$. Let $d_{ij} = \gcd(k_i, k_j)$. Then $d_{ij} = \min(k_i, k_j)$, by the divisor property. Applying the distributivity of tensor product over direct sum,

$$\begin{aligned} A \otimes_{\mathbb{Z}} A &\cong \left(\bigoplus_{j=1}^{N} \mathbb{Z}/k_j\mathbb{Z}\right) \otimes_{\mathbb{Z}} \left(\bigoplus_{i=1}^{N} \mathbb{Z}/k_i\mathbb{Z}\right) \cong \bigoplus_{i=1}^{N} \left(\left(\bigoplus_{j=1}^{N} \mathbb{Z}/k_j\mathbb{Z}\right) \otimes_{\mathbb{Z}} \mathbb{Z}/k_i\mathbb{Z}\right) \\ &\cong \bigoplus_{i=1}^{N} \left(\bigoplus_{j=1}^{N} (\mathbb{Z}/k_j\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/k_i\mathbb{Z})\right) \cong \bigoplus_{i,j} \mathbb{Z}/k_j\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/k_i\mathbb{Z} \cong \bigoplus_{i,j} \mathbb{Z}/d_{ij}\mathbb{Z} \end{aligned}$$

This direct sum can only be trivial if each each summand $\mathbb{Z}/d_{ij}\mathbb{Z}$ is trivial, that is, if each $d_{ij} = 1$. But no d_{ij} can be 1, since $d_{ij} = \min(k_i, k_j)$ and $k_i, k_j \geq 2$. Hence $A \otimes_{\mathbb{Z}} A$ is nontrivial. □

Proposition 0.7 (Exercise 2a)**.** *Let A be a nonzero finitely generated abelian group. Then $A \otimes_{\mathbb{Z}} A \neq 0$.*

Proof. Using the distributive property of the tensor product over direct sum, we get

$$\begin{aligned} A \otimes_{\mathbb{Z}} A &= (A_{\text{free}} \oplus A_{\text{tor}}) \otimes_{\mathbb{Z}} (A_{\text{free}} \oplus A_{\text{tor}}) \\ &\cong (A_{\text{free}} \otimes_{\mathbb{Z}} A_{\text{free}}) \oplus (A_{\text{tor}} \otimes_{\mathbb{Z}} A_{\text{free}}) \oplus (A_{\text{free}} \otimes_{\mathbb{Z}} A_{\text{tor}}) \oplus (A_{\text{tor}} \otimes_{\mathbb{Z}} A_{\text{tor}}) \end{aligned}$$

Note that a direct sum is trivial only if each direct summand is trivial. If A is free, then $A \otimes_{\mathbb{Z}} A$ is free of with rank equal to the square of the rank of A by Corollary 2.4 (Lang), so $A \otimes_{\mathbb{Z}} A$ is not zero. So we can assume that A is not free, that is, $A_{\text{tor}} \neq 0$. Then, by the previous lemma, $A_{\text{tor}} \otimes_{\mathbb{Z}} A_{\text{tor}} \neq 0$, so one summand is nontrivial, hence $A \otimes_{\mathbb{Z}} A \neq 0$. □

Proposition 0.8 (Exercise 2b)**.** $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$.

Proof. It suffices to show that for $\overline{x}, \overline{y} \in \mathbb{Q}/\mathbb{Z}$, we have $\overline{x} \otimes \overline{y} = 0$, since elements of this form generate $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$. Let $\overline{x} = x + \mathbb{Z}$ and $\overline{y} = y + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$, where $x, y \in \mathbb{Q}$. There exists $n \in \mathbb{Z}$ so that $ny \in \mathbb{Z}$, so then $n\overline{y} = ny + \mathbb{Z} = 0$. Then

$$\overline{x} \otimes \overline{y} = \left(\frac{n}{n}\overline{x}\right) \otimes \overline{y} = \left(\frac{x}{n} + \mathbb{Z}\right) \otimes (ny + \mathbb{Z}) = \left(\frac{x}{n} + \mathbb{Z}\right) \otimes 0 = 0$$

□

Proposition 0.9 (Exercise 2c)**.** *The tensor functor is not always left exact.*

Proof. Consider the exact sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ \phi\ } \mathbb{Z}$$

where $\phi(x) = 2x$. Considering $\mathbb{Z}/2\mathbb{Z}$ as a $\mathbb{Z}$-module, we get an induced sequence

$$0 \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\widetilde{\phi}} \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$$

where $\widetilde{\phi}(x \otimes y) = \phi(x) \otimes y = (2x) \otimes y = x \otimes (2y) = x \otimes 0 = 0$, so $\widetilde{\phi}$ is the trivial map. We know that $\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, so $\widetilde{\phi}$ is the trivial endomorphism of $\mathbb{Z}/2\mathbb{Z}$, which is not injective. Thus the induced sequence is not left exact. □

Proposition 0.10 (Chapter 3, Exercise 15a, the Five Lemma)**.** *Consider the following commutative diagram of R-modules, where each row is exact:*

$$\begin{array}{ccccccccc} M_1 & \xrightarrow{\phi_1} & M_2 & \xrightarrow{\phi_2} & M_3 & \xrightarrow{\phi_3} & M_4 & \xrightarrow{\phi_4} & M_5 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow & & {\scriptstyle f_5}\downarrow \\ N_1 & \xrightarrow{\psi_1} & N_2 & \xrightarrow{\psi_2} & N_3 & \xrightarrow{\psi_3} & N_4 & \xrightarrow{\psi_5} & N_5 \end{array}$$

Suppose that f_1 is surjective and f_2, f_4 are injective. Then f_3 is injective. (Note: We don't need M_5, N_5 and their maps.)

Proof. Let $x \in \ker f_3$. Then

$$\begin{aligned} x \in \ker f_3 &\implies f_3(x) = 0 \implies \psi_3 f_3(x) = 0 \\ \psi_3 f_3 = f_4\phi_3 &\implies f_4\phi_3(x) = 0 \implies \phi_3(x) \in \ker f_4 \\ f_4 \text{ injective} &\implies \phi_3(x) = 0 \implies x \in \ker \phi_3 \\ \ker \phi_3 = \operatorname{im} \phi_2 &\implies \exists y \in M_2, \phi_2(y) = x \\ f_3\phi_2 = \psi_2 f_2 &\implies f_3\phi_2(y) = 0 \implies \psi_2 f_2(y) = 0 \implies f_2(y) \in \ker \psi_2 \\ \ker \psi_2 = \operatorname{im} \psi_1 &\implies f_2(y) \in \operatorname{im} \psi_1 \implies \exists a \in N_1, \psi_1(a) = f_2(y) \\ f_1 \text{ surjective} &\implies \exists z \in M_1, f_1(z) = a \implies \psi_1 f_1(z) = \psi_1(a) = f_2(y) \\ \psi_1 f_1 = f_2\phi_1 &\implies f_2\phi_1(z) = f_2(y) \\ f_2 \text{ injective} &\implies \phi_1(z) = y \implies \phi_2\phi_1(z) = \phi_2(y) = x \\ \operatorname{im} \phi_1 = \ker \phi_2 &\implies \phi_2\phi_1 = 0 \implies \phi_2\phi_1(z) = x = 0 \end{aligned}$$

Thus $\ker f_3 = 0$, so f_3 is injective. □

Proposition 0.11 (Chapter 3, Exercise 15b, the Five Lemma)**.** *Consider the following commutative diagram of R-modules, where each row is exact:*

$$\begin{array}{ccccccccc} M_1 & \xrightarrow{\phi_1} & M_2 & \xrightarrow{\phi_2} & M_3 & \xrightarrow{\phi_3} & M_4 & \xrightarrow{\phi_4} & M_5 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow & & {\scriptstyle f_5}\downarrow \\ N_1 & \xrightarrow{\psi_1} & N_2 & \xrightarrow{\psi_2} & N_3 & \xrightarrow{\psi_3} & N_4 & \xrightarrow{\psi_4} & N_5 \end{array}$$

Suppose that f_5 is injective and f_2, f_4 are surjective. Then f_3 is surjective. (Note: We don't need M_1, M_2 for this.

Proof. Let $a \in N_3$. Then

$$\begin{aligned} f_4 \text{ surjective} &\implies \exists x \in M_4, f_4(x) = \psi_3(a) \\ \operatorname{im}\psi_3 = \ker\psi_4 &\implies \psi_3(a) \in \ker\psi_4 \\ f_5\phi_4 = \psi_4 f_4 &\implies f_5\phi_4(x) = \psi_4 f_4(x) = \psi_4\psi_3(a) = 0 \\ f_5 \text{ injective} &\implies \phi_4(x) = 0 \implies x \in \ker\phi_4 \\ \ker\phi_4 = \operatorname{im}\phi_3 &\implies \exists y \in M_3, \phi_3(y) = x \\ f_4\phi_3 = \psi_3 f_3 &\implies \psi_3 f_3(y) = f_4\phi_3(y) = f_4(x) = \psi_3(a) \\ \psi_3 \text{ is } R\text{-linear} &\implies \psi_3(f_3(y) - a) = 0 \implies f_3(y) - a \in \ker\psi_3 \\ \ker\psi_2 = \operatorname{im}\psi_2 &\implies \exists b \in N_2, \psi_2(b) = f_3(y) - a \\ f_2 \text{ surjective} &\implies \exists z \in M_2, f_2(z) = b \\ f_3\phi_2 = \psi_2 f_2 &\implies f_3\phi_2(z) = \psi_2 f_2(z) = \psi_2(b) = f_3(y) - a \\ f_3 \text{ is } R\text{-linear} &\implies a = f_3(y) - f_3\phi_2(z) = f_3(y - \phi_2(z)) \implies a \in \operatorname{im} f_3 \end{aligned}$$

Thus f_3 is surjective. □

Proposition 0.12 (Chapter 3, Exercise 15c, part one)**.** *Suppose we have a commutative diagram with exact rows,*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \end{array}$$

Suppose that f, h are isomorphisms. Then g is an isomorphism.

Proof. By adding the tacit isomorphisms $0 \to 0$ on both ends, we have a diagram that satisfies the hypotheses of parts (a) and (b), since f, h are bijective by hypothesis. Thus by part (a), g is injective, and by part (b), g is surjective. Thus it is an isomorphism. □

In the previous proposition, we assumed that there was a homomorphism $g : M \to N$ that makes two squares commute. However, if one has a diagram with exact rows of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & & & {\scriptstyle h}\downarrow & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \end{array}$$

then M and N need not be isomorphic. For example,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{1\mapsto 2} & \mathbb{Z} & \xrightarrow{1\mapsto 1} & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \mathrm{Id}} & & & & \Big\downarrow{\scriptstyle \mathrm{Id}} & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{1\mapsto(1,0)} & \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} & \xrightarrow[(0,1)\mapsto 1]{(1,0)\mapsto 0} & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0
\end{array}$$

The vertical arrows are isomorphisms. The top row is exact because the image of $1 \mapsto 2$ is $2\mathbb{Z}$, which gets mapped to zero under the projection $\mathbb{Z} \mapsto \mathbb{Z}/2\mathbb{Z}$. The bottom row is exact because the image of $(1 \mapsto (1,0)$ is $\mathbb{Z} \oplus 0$, which gets mapped to zero under $(1,0) \mapsto 0$. However, $\mathbb{Z}$ is not isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, because the former is torsion free, while the latter has one element of order 2.

Lemma 0.13 (for Chapter XX, Exercise 1)**.** *Let*

$$\dots \longrightarrow E^{i+1} \xrightarrow{d^{i+1}} E^i \xrightarrow{d^i} E^{i-1} \longrightarrow \dots$$

be a sequence of R-modules and R-module homomorphisms. Suppose there exist R-module homomorphisms $h^i : E^i \to E^{i+1}$ so that $d^{i+1} \circ h^i + h^{i-1} \circ d^i = \mathrm{Id}_{E^i}$, and $d^{i+1} \circ d^i = 0$ for all i. Then the sequence is exact.

Proof. The hypothesis that $d^{i+1} \circ d^i = 0$ tells us that $\operatorname{im} d^{i+1} \subset \ker d^{i+1}$ for each i. To get the reverse inclusion, let $x \in \ker d^i$. Then $h^{i-1} \circ d^i = 0$ so

$$(d^{i+1} \circ h^i + h^{i-1} \circ d^i)(x) = \mathrm{Id}_{E^i}(x) = x \implies d^{i+1}(h^i(x)) = x \implies x \in \operatorname{im} d^{i+1}$$

Hence $\ker d^i \subset \operatorname{im} d^{i+1}$. Thus the sequence is exact. $\square$

Proposition 0.14 (Chapter XX, Exercise 1)**.** *Let S be a set. The standard complex obtained from S is exact, and hence is a resolution of $\mathbb{Z}$.*

Proof. Fix $z \in S$. Define $h_z : E^i \to E^{i+1}$ by $h(x_0, \dots, x_i) = (z, x_0, \dots, x_i)$. For convenience, we'll just write h instead of h_z. Note that h is a homomorphism. We claim that $d^{i+1} \circ h + h \circ d^i = \mathrm{Id}_{E^i}$. It suffices to show that it acts as the identity on the generators.

$$\begin{aligned}
(d^{i+1} \circ h + h \circ d^i)(x_0, \dots, x_i) &= d^{i+1} \circ h(x_0, \dots, x_i) + h \circ d^i(x_0, \dots, x_i) \\
&= d^{i+1}(z, x_0, \dots, x_i) + h\left(\sum_{j=0}^{i} (-1)^j (x_0, \dots, \widehat{x}_j, \dots, x_i)\right) \\
&= (x_0, \dots, x_i) + \sum_{j=0}^{i} (-1)^{j+1} (z, x_0, \dots, \widehat{x}_j, \dots, x_i) + \sum_{j=0}^{i} (-1)^j (z, x_0, \dots, \widehat{x}_j, \dots, x_i) \\
&= (x_0, \dots, x_i) - \sum_{j=0}^{i} (-1)^j (z, x_0, \dots, \widehat{x}_j, \dots, x_i) + \sum_{j=0}^{i} (-1)^j (z, x_0, \dots, \widehat{x}_j, \dots, x_i) \\
&= (x_0, \dots, x_i)
\end{aligned}$$

Thus $d^{i+1} \circ h + h \circ d^i = \mathrm{Id}_{E^i}$. We also claim that $d^i \circ d^{i+1} = 0$. As before, it suffices to show that $d_i \circ d_{i+1} = 0$ for generators of E_{i+1}.

$$d^i \circ d^{i+1}(x_0, \dots, x_{i+1}) = d^i \left(\sum_{j=0}^{i+1} (-1)^j (x_0, \dots, \widehat{x}_j, \dots, x_{i+1}) \right) = \sum_{j=0}^{i+1} (-1)^j d^i (x_0, \dots, \widehat{x}_j, \dots, x_{i+1})$$

To apply d^i to $(x_0, \dots, \widehat{x}_j, \dots, x_{i+1})$, we have to be careful about the indices, because for $k > j$ the power of (-1) no longer matches the subscript on the x_i's.

$$\begin{aligned} d^i(x_0, \dots, \widehat{x}_j, \dots, x_{i+1}) = \sum_{k<j} &(-1)^k (x_0, \dots, \widehat{x}_k, \dots, \widehat{x}_j, \dots, x_{i+1}) \\ &+ \sum_{k>j} (-1)^{k-1} (x_0, \dots, \widehat{x}_j, \dots, \widehat{x}_k, \dots, x_{i+1}) \end{aligned}$$

Now we can plug this in to continue the calculation.

$$\begin{aligned} d^i \circ d^{i+1}(x_0, \dots, x_{i+1}) &= \sum_{j=0}^{i+1} (-1)^j \left(\sum_{k<j} (-1)^k (x_0, \dots, \widehat{x}_k, \dots, \widehat{x}_j, \dots, x_{i+1}) \right. \\ &\qquad \left. + \sum_{k>j} (-1)^{k-1} (x_0, \dots, \widehat{x}_j, \dots, \widehat{x}_k, \dots, x_{i+1}) \right) \\ &= \sum_{k<j} (-1)^{j+k} (x_0, \dots, \widehat{x}_k, \dots, \widehat{x}_j, \dots, x_{i+1}) \\ &\qquad + \sum_{j<k} (-1)^{j+k-1} (x_0, \dots, \widehat{x}_j, \dots, \widehat{x}_k, \dots, x_{i+1}) \end{aligned}$$

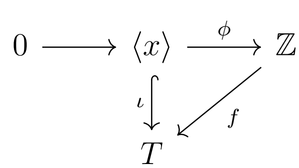
Now notice that j, k are dummy variables, so in the second summation we can interchange their roles, and pull out a (-1), to see that the two summations exactly cancel out.

$$\sum_{k<j} (-1)^{j+k} (x_0, \dots, \widehat{x}_k, \dots, \widehat{x}_j, \dots, x_{i+1}) - \sum_{k<j} (-1)^{k+j} (x_0, \dots, \widehat{x}_k, \dots, \widehat{x}_j, \dots, x_{i+1}) = 0$$

Thus we have shown that $d^{i+1} \circ h + h \circ d^i = \mathrm{Id}_{E^i}$ and $d_i \circ d_{i+1} = 0$, so applying the previous lemma, the sequence is exact. □

Proposition 0.15 (Exercise 6a)**.** *Let T be an injective object in the category of abelian groups. Then T is divisible.*

Proof. Suppose that T is not divisible. Then there exists $x \in T$ and $n \in \mathbb{N}$ so that $x \notin nT$. Then $\langle x \rangle$ is a cyclic subgroup of T, and we have the inclusion homomorphism $\iota : \langle x \rangle \hookrightarrow T$. First suppose that $\langle x \rangle$ is infinite. Then the map $\phi : \langle x \rangle \to \mathbb{Z}$ defined by $x \mapsto n$ is injective, so by injectivity of T there exists $f : \mathbb{Z} \to T$ so that the following diagram commutes.



This implies that

$$nf(1) = f(n) = f\phi(x) = \iota(x) = x \implies x \in nT$$

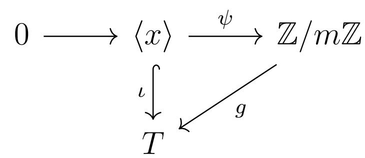
Now suppose that $\langle x \rangle$ is finite, with order $|x| \in \mathbb{N}$. Let $m = n|x|$, and define $\psi : \langle x \rangle \to \mathbb{Z}/m\mathbb{Z}$ by $kx \mapsto kn + m\mathbb{Z}$. We check that ψ is well-defined. If $kx = k'x$, then

$$\begin{aligned} kx = k'x &\implies |x|\big|(k-k') \implies m|(k-k')n \implies kn - k'n \in m\mathbb{Z} \\ &\implies kn + m\mathbb{Z} = k'n + m\mathbb{Z} \implies \psi(kx) = \psi(k'x) \end{aligned}$$

Thus ψ is well-defined. We claim that ψ is injective.

$$kx \in \ker \psi \implies kn + m\mathbb{Z} = m\mathbb{Z} \implies kn \in m\mathbb{Z} \implies m|kn \implies |x|\big|k \implies kx = 0$$

Thus ψ is injective. By injectivity of T, there exists $g : \mathbb{Z}/m\mathbb{Z} \to T$ so that the following diagram commutes.



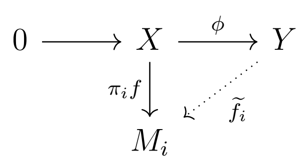
This implies that

$$ng(1 + m\mathbb{Z}) = g(n + m\mathbb{Z}) = g\psi(x) = \iota(x) = x \implies x \in nT$$

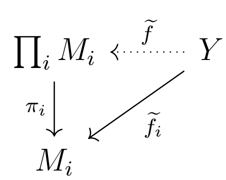
We constructed x so that $x \notin nT$, but we showed that in either of two cases, $x \in nT$. This is a contradiction, so we conclude that T is divisible. □

Proposition 0.16 (Exercise 6b)**.** *A direct product of injective modules is injective.*

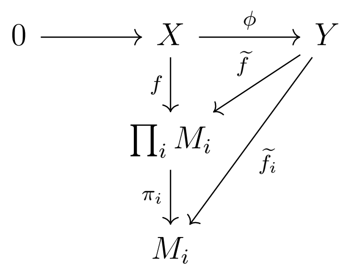
Proof. Let R be a commutative ring and $\{M_i\}_{i \in I}$ a collection of R-modules. Let $M = \prod_{i \in I} M_i$ and let $\pi_i : \prod_i M_i \to M_i$ be the projection onto the ith factor. Let X, Y be R-modules and $\phi : X \to Y$ be an injective homomorphism, and let $f : X \to \prod_i M_i$ be a homomorphism. Then we have homomorphisms $\pi_i f : X \to M_i$, so by injectivity of M_i, there exists a homomorphism $\widetilde{f}_i : Y \to M_i$ making the following diagram commute:



Then, by the universal property of the direct product, there is a unique morphism $\widetilde{f} : Y \to \prod_i M_i$ so that $\widetilde{f}\pi_i = \widetilde{f}_i$. In particular, $\widetilde{f}$ is the map $y \mapsto (\widetilde{f}_i(y))$. That is, the following diagram commutes.



Putting these diagrams together, we get the following diagram:



We just need to check commutativity of the $X, Y, \prod_i M_i$ triangle. Using the commutativity of the other triangles, for $x \in X$ we have

$$\widetilde{f}\phi(x) = (\widetilde{f}_i\phi(x)) = (\pi_i f(x)) = f(x)$$

Thus $\widetilde{f}\phi = f$, so the required triangle commutes. Hence $\prod_i M_i$ is injective. $\square$